Week 6: The Mapping Cylinder

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Contents

1 Instructions

Please complete all exercises. There are no exercises in §5.

2 Spaces Under A

Definition 1 *Let A be a space. A* ***space under*** *A is a map $f : A \to X$. If $f : A \to X$ and $g : A \to Y$ are spaces under A, then a map $\alpha : X \to Y$ is said to be a* ***map under*** *A if it makes the following triangle commute*

$$\begin{array}{ccc} & A & \\ {}^{f}\swarrow & & \searrow^{g} \\ X & \xrightarrow{\alpha} & Y. \end{array} \tag{2.1}$$

Identity functions are maps under A. Moreover there is a well-defined composition of maps under A. Thus the spaces and maps under A form a category which we denote A/Top and call the ***category of spaces under*** *A.* $\square$

We call a space under A an **underspace** when repeated reference to A becomes clunky, and write (X, f) to denote a given underspace $f : A \to X$. The map f itself is said to be the **structure map** of (X, f). We sometimes write $\alpha : X \overset{A}{\to} Y$ to denote a morphism in A/Top.

Example 2.1

1. *If $A = \emptyset$, then $\emptyset/Top$ isomorphic to Top.*

2. *If $A = *$, then $*/Top$ isomorphic to Top_*.*

3. *If $A = S^0$, then S^0/Top is the category of bipointed spaces. The objects are topological spaces with two distinguished basepoints, and maps under S^0 must preserve both basepoints.* □

There is a forgetful functor

$$A/Top \overset{U}{\to} Top, \qquad (X, f) \mapsto X \tag{2.2}$$

which sends an underspace $f : A \to X$ to the space X. This functor has a left adjoint K

$$Top \underset{U}{\overset{K}{\rightleftarrows}} A/Top, \qquad K \dashv U. \tag{2.3}$$

Exercise 2.1 Construct the functor $Top \overset{K}{\to} A/Top$. The property it must satisfy is the following: for a space M and an underspace (X, f) there is a bijection

$$A/Top\big(K(M), (X, f)\big) \cong Top\big(M, U(X, f)\big) \tag{2.4}$$

which is natural in both variables. □

With a little help from the adjunction 2.3 the following can now be made rigourous.

Proposition 2.1 *The category A/Top has all limits and colimits.* ■

The canonical reference for the category theoretical details is Borceaux's book [1] §2 and §4. As usual, in this course I will only ask you to understand the objects we will need.

Exercise 2.2 Identify the products, coproducts in the category A/Top. (You do not need to rigourously construct these objects. Just observe the correct construction and write it down.) □

Next we would like to define a notion of homotopy in the category A/Top. Although we could define internal cylinders in A/Top and define homotopy in terms of these objects, in practice they are difficult to work with. Thus we will prefer a more direct approach.

Definition 2 *Let (X, f), (Y, g) be spaces under A. We say that a homotopy $H : X \times I \to Y$ is a **homotopy under** A if at each time $t \in I$ the map $x \mapsto H_t(x)$ is a map under A*

$$\begin{array}{ccc} & A & \\ {}^{f}\swarrow & & \searrow^{g} \\ X & \xrightarrow{H_t} & Y. \end{array} \tag{2.5}$$

□

Thus H is a homotopy under A if it satisfies

$$H_t(f(a)) = g(a), \qquad \forall t \in I, a \in A. \tag{2.6}$$

The intuition is clearest when f, g are subspace inclusions.

Proposition 2.2 *Homotopy under A is an equivalence relation which is compatible with composition.* $\blacksquare$

We have all the usual notions of homotopy equivalence under A, left-/right- homotopy inverse under A, etc... In fact homotopy under A gives an elegant way to express many familiar ideas.

Exercise 2.3 Let $f : A \to X$ be a space under A. Show that A is a strong deformation retract of X if and only if the map $f : (A, id_A) \stackrel{A}{\to} (X, f)$ has a left homotopy inverse in the category A/Top. □

3 The Mapping Cylinder

Let $f : X \to Y$ be a map. Over the last few exercise sheets we encountered both the *mapping cylinder* M_f and *mapping cone* C_f of f. There were many similarities between the two constructions, both intimately linked to the theory of cofibrations. It is the purpose of this week's exercises to explore the connection more fully. For ease *we will work throughout in the unpointed category.* There are analogous constructions in the pointed category which will be discussed at the end of these notes. By restricting to well-pointed spaces we shall be able to transfer statements between the two categories with ease.

Definition 3 *The unreduced **mapping cylinder** of a map $f : X \to Y$ is the space $\widetilde{M}_f$ defined by the pushout*

$$\begin{array}{ccc} X & \xrightarrow{in_0} & X \times I \\ {\scriptstyle f}\downarrow & \lrcorner & \downarrow \\ Y & \xrightarrow[l_f]{} & \widetilde{M}_f. \end{array} \tag{3.1}$$

In particular we will understand

$$\widetilde{M}_f = \frac{Y \sqcup X \times I}{[f(x) \sim (x, 0)]}. \tag{3.2}$$

We write

$$l_f : Y \hookrightarrow \widetilde{M}_f \tag{3.3}$$

for the canonical map. □

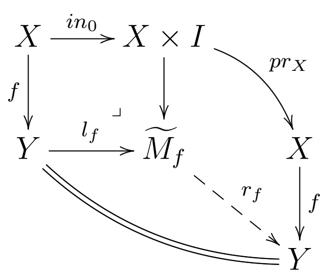
Notation: The tilde on $\widetilde{M}_f$ is to distinguish it from the *reduced mapping cylinder*, i.e. the construction made in the pointed category by replacing $X \times I$ with $X \wedge I_+$. □

Example 3.1

1. *The mapping cylinder of* $X \to *$ *is the (unreduced) cone over* X*.*
2. *The mapping cylinder of* $* \to Y$ *is the space* $Y \cup I$*, which is obtained from* Y *by 'growing a whisker' over the basepoint.*
3. *The mapping cylinder of* id_X *is the cylinder* $X \times I$*.* □

Now the inclusion $X \hookrightarrow X \times I$ is both a closed cofibration and a homotopy equivalence, and this implies that so is the map $l_f : Y \hookrightarrow \widetilde{M}_f$. This is a consequence of Proposition 6.3 of *Fibrations II*. Moreover we know from *Cofibrations* Exercise 3.3 that this implies that Y is a strong deformation retract of $\widetilde{M}_f$. Your first exercise this week will be to check these details explicitly.

Use the diagram



(3.4)

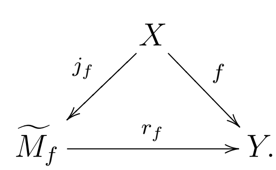
to define a map $r_f : \widetilde{M}_f \to Y$.

Exercise 3.1 Write down an explicit homotopy to show that r_f is a homotopy equivalence under Y. □

Now define a map $j_f : X \to \widetilde{M}_f$ as the composite

$$j_f : X \xrightarrow{in_1} X \times I \to \widetilde{M}_f, \qquad x \mapsto (x, 1). \tag{3.5}$$

Then this us gives a factorisation of f through its mapping cylinder in a strictly commutative diagram



(3.6)

Moreover there is a canonical homotopy

$$l_f \circ f \simeq j_f. \tag{3.7}$$

Note, however, that this is not a homotopy *under* X.

Exercise 3.2 Show that $j_f : X \to \widetilde{M_f}$ is a closed cofibration. □

Thus the following makes sense.

Definition 4 *Let $f : X \to Y$ be a map. We call the cofibration $j_f : X \hookrightarrow \widetilde{M_f}$ the result of* ***converting f into a cofibration****.* □

The idea is that the mapping cylinder gives a *functorial* way of replacing a map with a *pointwise equivalent* cofibration. We won't be interested in functorality at this stage, but will rather try to unravel the second statement.

To explain ourselves consider diagram (3.6), which displays r_f as a map under X. You showed in 3.1 that r_f is a homotopy equivalence, but you'll notice that you did not show there that it is a homotopy equivalence under X. This is what we mean by *pointwise equivalence.* It would be much preferable if Y and $\widetilde{M_f}$ were homotopy equivalent under X.

Is it possible that these spaces can be homotopy equivalent in this way? And why would this be desirable? As it turns out, the answer to these questions is found in that of another question.

> What if f is already a cofibration? Why replace something which doesn't need to be replaced?

In the exercises of the next section you will answer this question in much generality. A consequence of your work will be that if f is already a cofibration, then r_f in (3.6) is actually a homotopy equivalence under X. A conclusion will be that the mapping cone construction is the unique way to turn f into a fibration, up to homotopy equivalence under X.

4 Cofiber Homotopy Equivalences

Theorem 4.1 *Let*

$$\begin{array}{ccc} & A & \\ {}^{f}\swarrow & & \searrow^{g} \\ X & \xrightarrow{\alpha} & Y. \end{array} \tag{4.1}$$

be a diagram of spaces under A in which α is an ordinary homotopy equivalence. If f, g are both cofibrations, then α is a homotopy equivalence under A.

The remainder of the exercises will be devoted to proving this theorem.

Exercise 4.1 With the assumptions and notations of Theorem 4.1, show that there is a map $\beta : Y \to X$ which is both a map under A and an ordinary homotopy inverse to α. □

Now, we claim that Theorem 4.1 is equivalent to the following statement.

> *If $f : A \to X$ is a cofibration and*
>
> $$\begin{array}{ccc} & A & \\ {}^{f}\swarrow & & \searrow^{f} \\ X & \xrightarrow{\alpha} & X. \end{array} \tag{4.2}$$
>
> *is a map under A such that $\alpha \simeq id_X$ in Top, then α has a left homotopy inverse in A/Top.*

One direction of the equivalence is clear. To see how the last statement implies 4.1 we use Exercise 4.1. Composing (5.3) with the map β constructed there we find ourselves in the situation of (4.2). Then switching the rôles of α and β we apply the statement again to see that it indeed implies Theorem 4.1.

You should pause a second to check through the details and convince yourself that the last paragraph makes sense because we'll use the reformulation to prove 3.6. Before this, though, we need a lemma to help organise information under A.

Lemma 4.2 *Let $f : A \hookrightarrow X$ be a cofibration and $\varphi, \theta : X \to Y$ maps under A as in the diagram*

$$\begin{array}{ccc} & A & \\ {\scriptstyle f}\swarrow & & \searrow{\scriptstyle g} \\ X & \underset{\theta}{\overset{\varphi}{\longrightarrow}} & Y. \end{array} \tag{4.3}$$

Suppose that $G : \varphi \simeq \theta$ is an ordinary homotopy and that $\Psi : A \times I \times I \to Y$ is a track homotopy $\psi : Gf \sim K$. Then there is a homotopy $\widetilde{G} : \varphi \simeq \theta$ such that $\widetilde{G}f = K$.

Notice that $K : A \times I \to Y$ is an ordinary homotopy $g = \varphi f \simeq \theta f = g$.

Exercise 4.2 Use the fact that

$$X \times 0 \cup A \times I \cup X \times 1 \hookrightarrow X \times I \tag{4.4}$$

is a cofibration to prove Lemma 4.2. □

We'll use the lemma in a second. First we need to set up a little more. Keeping the notation of 4.2 begin by choosing a homotopy

$$H : \alpha \simeq id_X. \tag{4.5}$$

Now use the fact that f is a cofibration to find a retraction[1] $r : X \times I \to A \times I \cup X \times 0$ and use it to define a homotopy J as the composite

$$J : X \times I \xrightarrow{r} (A \times I) \cup (X \times 0) \xrightarrow{(Hf) \cup id_X} X. \tag{4.6}$$

Then $J_0 = id_X$ and $J_t f = H_t f$. Put

$$\beta = J_1 : X \to X. \tag{4.7}$$

Notice that $\beta f = J_1 f = H_1 f = \alpha f = f$, so this is a map under A.

Now we're ready. In the next exercise we'll want to apply Lemma 4.2 so we'll set things up to make use of its notation by letting

1. $G = -J\alpha + H$
2. $\varphi = \beta\alpha$
3. $\theta = id_X$.

Exercise 4.3 Assemble the pieces and prove Theorem 4.1. □

[1] See *Cofibrations* Theorem 1.3

5 Applications

There are two basic applications for the mapping cylinder which we would like to discuss. Both are related, but will take us in slightly different directions.

5.1 Some Consequences of Theorem 5.3

You were asked to prove the following using other methods in an earlier exercise sheet. It is now an easy corollary of 5.3.

Proposition 5.1 *If $j : A \hookrightarrow X$ is both a cofibration and a homotopy equivalence, then A is a strong deformation retract of X.*

Proof This follows from Exercise 2.3. $\blacksquare$

Corollary 5.2 *Let $f : X \xrightarrow{\simeq} Y$ be a homotopy equivalence. Then X is a strong deformation retract of $\widetilde{M_f}$. In particular, two spaces are homotopy equivalent if and only if they are deformation retracts of the same space.*

Proof This follows from Exercise 3.1. $\blacksquare$

Another application we have for Theorem 5.3 is to deciding how different choices of basepoint affect the pointed homotopy type of an unbased space.

Proposition 5.3 *Let X be a space and $x_0, x_1 \in X$ points such that each inclusion $x_0 \hookrightarrow X$ and $x_1 \hookrightarrow X$ is a cofibration. Assume that there is a path $l : I \to X$ with $l(i) = x_i$, $i = 0, 1$. Then $(X, x_0) \simeq (X, x_1)$ as pointed spaces.*

Proof Apply the HEP to the pair (id_X, l) to find a homotopy $H : X \times I \to X$ with $H_0 = id_X$ and $H_t(x_0) = l(t)$. Set $\alpha = H_1$. Then $\alpha(x_0) = x_1$, and $\alpha \simeq id_X$ freely. By assumption the inclusions $x_0, x_1 \hookrightarrow X$ are cofibrations, so we can apply Theorem 4.1 to get the statement. $\blacksquare$

Corollary 5.4 *If X is a connected CW complex, then it has a well-defined pointed homotopy type.*

Proof The path component of a CW complex conicide with its connected components, sine each CW complex is locally contractible [2]. Moroever, if X is CW complex and $x \in X$ is any point, then it is known that X has a (possibly different) CW structure which has x as a vertex [2] pg. 67. In particular the inclusion of any point into X is a cofibration. Hence the statement follows from 5.3. $\blacksquare$

Remark The also statement is true also when X is replaced by a connected manifold. $\square$

5.2 Uniqueness of the Mapping Cylinder

Next we want to address the question raised earlier. How unique is the mapping cylinder construction? So assume that $f : X \to Y$ is a map, and we have constructed a commutative diagram

$$\begin{array}{ccc} & X & \\ {}^{k_f}\swarrow & & \searrow^{f} \\ N_f & \xrightarrow{s_f} & Y \end{array} \tag{5.1}$$

where N_f is some space, k_f is a cofibration and s_f is a homotopy equivalence. How does this compare to the mapping cylinder (3.6)?

We choose a homotopy inverse to s_f and consider the composite

$$\widetilde{\theta} = s_f^{-1} r_f : \widetilde{M_f} \to Y \to N_f. \tag{5.2}$$

Since r_f and s_f are homotopy equivalences, so is $\widetilde{\theta}$. Next we use the fact that $j_f : X \hookrightarrow \widetilde{M_f}$ is a cofibration to replace $\widetilde{\theta}$ with a homotopic map θ satisfying $\theta j_f = k_f$. Then θ is a homotopy equivalent under X. Thus when we finally apply Theorem 5.3 we can conclude the following.

Proposition 5.5 *Up to homotopy equivalence under X, the mapping cylinder construction is the unique way to replace a map $f : X \to Y$ by a pointwise equivalence cofibration.* ∎

5.3 The Mapping Cylinder Again

Fix a map $f : X \to Y$. To discuss the first application that we have in mind we will need to recall the result of Exercise 3.2. Namely that there is a strictly commutative diagram

$$\begin{array}{ccc} & X & \\ {}^{j_f}\swarrow & & \searrow^{f} \\ M_f & \xrightarrow{r_f} & Y. \end{array} \tag{5.3}$$

in which j_f is a cofibration and r_f is a homotopy equivalence.

What we would like to draw special attention to is that j_f is a closed cofibration and in particular a closed embedding. This is quite nice since there are many constructions in topology of a more geometric nature that require the use of pairs of spaces. The mapping cylinder allows for such constructions to be understood for arbitrary maps, rather than just subspace inclusions.

For instance we can now make sense of a long exact sequence of cohomology groups when given an arbitrary map $f : X \to Y$. We simply define the relative groups in this case to be $H^*(M_f, X)$. This makes sense since X is embedded in M_f as a closed subspace and gives us the following

$$\begin{array}{ccccccccc} \ldots \longrightarrow & H^*(M_f, X) & \dashrightarrow & H^*Y & \xrightarrow{f^*} & H^*X & \longrightarrow & H^{*+1}(M_f, X) & \longrightarrow \ldots \\ & & {}_{\Delta}\searrow & {\scriptstyle r_f^*}\downarrow{\scriptstyle \cong} & \nearrow_{j_f^*} & & & & \\ & & & H^*M_f & & & & & \end{array} \tag{5.4}$$

The solid arrows here come from the long-exact sequence of the pair (M_f, X). The map r_f^* is an isomorphism, since r_f is a homotopy equivalence, and we define the dotted arrow using its inverse l_f^*. In this way the row becomes a long exact sequence. Here it doesn't matter that $l_f \circ f \simeq j_f$ is not a homotopy under X because for cohomology its existence alone is enough to give us

$$j_f^* = (l_f f)^* = f^* l_f^* \tag{5.5}$$

which is what we need to get exactness.

We'll discuss the naturality of these sequences at a later point. For what we need now we'll need to recall that the unreduced mapping cylinder $\widetilde{C}_f$ of a map $f : X \to Y$ is the pushout space in the next square

$$\begin{array}{ccc} X & \xrightarrow{in_0} & \widetilde{C}X \\ {\scriptstyle f}\downarrow & \lrcorner & \downarrow \\ Y & \longrightarrow & \widetilde{C}_f. \end{array} \tag{5.6}$$

We check easily the relation between the mapping cone and mapping cylinder of f.

$$\widetilde{M}_f / j_f(X) \cong \widetilde{C}_f. \tag{5.7}$$

That is $\widetilde{C}_f$ is the cofiber of $j_f : X \hookrightarrow \widetilde{M}_f$.

Lemma 5.6 *The collapse map $\widetilde{M}_f \to \widetilde{M}_f / j_f(X) \cong \widetilde{C}_f$ induces an isomorphism*

$$\widetilde{H}^*(\widetilde{C}_f) \cong H^*(\widetilde{C}_f, *) \xrightarrow{\cong} H^*(\widetilde{M}_f, X) \tag{5.8}$$

Proof Set

$$\widetilde{C}'X = \frac{X \times [1/3, 1]}{X \times 1} \subseteq \widetilde{C}_f, \tag{5.9}$$

Then $\widetilde{C}'X$ is contractible and we check easily that the quotient map is a homotopy equivalence of pairs $(\widetilde{C}_f, \widetilde{C}'X) \xrightarrow{\simeq} (\widetilde{C}_f, *)$. This map induces the isomorphism on the left-hand side of the next diagram.

$$\begin{array}{ccc} H^*(\widetilde{C}_f, *) & \dashrightarrow & H^*(\widetilde{M}_f, X) \\ \cong\downarrow & & \cong\downarrow{\scriptstyle excision} \\ H^*\left(\widetilde{C}_f, \widetilde{C}'X\right) & \xrightarrow[excision]{\cong} & H^*\big(Y \cup_f (X \times [0, 2/3]), X \times [1/3, 2/3]\big) \end{array} \tag{5.10}$$

The dotted arrow on the top of the diagram is an isomorphism and is exactly the map in question. ∎

Corollary 5.7 *If $j : A \hookrightarrow X$ is a cofibration, then the quotient map $X \to X/A$ induces an isomorphism*

$$\widetilde{H}^*(X/A) \cong H^*(X, A). \tag{5.11}$$

Proof Since j is a cofibration X/A is almost well-pointed. Similarly $\widetilde{C}_j$ is well-pointed when we base it at the cone point. Then the map $\widetilde{C}_j \to \widetilde{C}_j/\widetilde{C}A \cong X/A$ is a homotopy equivalence and based map, so according to Theorem 4.1 is a pointed homotopy equivalence. We also know that $r_j : \widetilde{M}_j \to X$ is a homotopy equivalence under A, which implies in particular that it induces a homotopy equivalence of pairs $(\widetilde{M}_j, A) \xrightarrow{\simeq} (X, A)$. We put these observations together with Lemma 5.6 to get the isomorphisms in the next diagram

$$\begin{array}{ccc} H^*(X/A, *) & \xrightarrow{\cong} & H^*(\widetilde{C}_j, *) \\ \downarrow & & \downarrow \cong \\ H^*(X, A) & \underset{\cong}{\xrightarrow{r_j^*}} & H^*(\widetilde{M}_j, A) \end{array} \tag{5.12}$$

and the conclusion follows. ∎

Thus when $j : A \hookrightarrow X$ is a cofibration there is a long exact sequence of abelian groups

$$\ldots \to \widetilde{H}^{n-1}A \to \widetilde{H}^n X/A \xrightarrow{q^*} \widetilde{H}^n X \xrightarrow{j^*} \widetilde{H}^n A \xrightarrow{\partial} \widetilde{H}^{n+1}X/A \xrightarrow{q^*} \widetilde{H}^{n+1}X \xrightarrow{j^*} \ldots \tag{5.13}$$

Compare this to Hatcher's notion of a *good pair* in [3] Th. 2.13, pg. 114. Not every cofibration $j : A \hookrightarrow X$ defines a good pair (X, A), but the fact that (X, A) is homotopy equivalent to (M_j, A) as pairs, and (M_j, A) is good is enough for us to extend Hatcher's results.

On the other hand, if $f : X \to Y$ is any map, then there is the long exact sequence (5.4), which under 5.6 becomes

$$\ldots \to \widetilde{H}^{n-1}X \to \widetilde{H}^n\widetilde{C}_f \to \widetilde{H}^n Y \xrightarrow{f^*} \widetilde{H}^n X \to \widetilde{H}^{n+1}\widetilde{C}_f \to \widetilde{H}^{n+1}Y \to \ldots \tag{5.14}$$

This can be especially useful when we can identify $\widetilde{C}_f$ explicitly. For example it shows directly how the attaching maps of a CW complex influence its cohomology.

References

[1] F. Borceux, *Handbook of Categorical Algebra I: Basic Category Theory*, Cambridge University Press, 1994.

[2] R. Fritsch, R. Piccinini, *Cellular Structures in Topology*, Cambridge University Press, (1990).

[3] A. Hatcher, *Algebraic Topology*, Cambridge University Press, (2002). Available at http://pi.math.cornell.edu/ hatcher/AT/ATpage.html.